PERIODIC ORBITS

Non-autonomous ODE's $\dot{x} = f_0(t,x) + \varepsilon f_1(t,x)$, $x \in \mathbb{R}^n$

with $f_0(t,x)$, $f_1(t,x) \in C^1(\mathbb{R}\times\mathbb{R}^n; \mathbb{R}^n)$

T periodic in time: $f_i(t,x) = f_i(t+T, x)$ $\forall t, x$

Assume the unperturbed equation

$$\dot{x} = f_0(t,x)$$

has T-periodic solution $x_0(t)$

Q: Does the periodic solution persists, slightly deformed, $\forall \varepsilon \neq 0$, as a T-periodic solution of the complete system?
i.e. is there a periodic orbit of

$$\dot{x} = f_0(t,x) + \varepsilon f_1(t,x)$$

$\forall \varepsilon \neq 0$ small, close to $x_0(t)$?

Put the problem in functional setting and apply IFT
Look for a zero of

$$\mathcal{F}: C^1_T \times \mathbb{R} \longrightarrow C^0_T$$

$$(x(t), \varepsilon) \longrightarrow \mathcal{F}(x(t),\varepsilon) = \dot{x}(t) - f_0(t,x(t)) - \varepsilon f_1(t,x(t))$$

here $C^k_T = \{ x(t) \in C^k(\mathbb{R}): \quad x(t+T) = x(t) \ \forall t \}$

↑ incorporate properties of sol (periodicity) in functional space

$\cdot)$ $F(x_0(t), 0) = \dot{x}_0(t) - f_0(t, x_0(t)) = 0$

as $x_0(t)$ sol

$\cdot)$ $F \in \mathcal{C}^1(C^1_T \times \mathbb{R}, \mathcal{C}^0_T)$: $F = F_1 + F_2 + F_3$

$\to$ $F_1(x(t)) = \frac{d}{dt} x(t) = A\, x(t)$ linear op

If F_1 is bd, it is diff and $d F_1(x(t))[h(t)] = A\, h(t)$

$\| A\, h(t) \|_{C^0} = \| \frac{d}{dt} h(t) \|_{\mathcal{C}^1} \leq \| h(t) \|_{\mathcal{C}^1}$ ✓

$\to$ $F_2(x(t)) = f_0(t, x(t))$ Nemitski operator !

we know that since $f_0 \in \mathcal{C}^1$ and $x(t) \in \mathcal{C}^1_T$, then

$f_0(t, x(t)) \in C^1_T$ and $d_x F_2(x(t))[h(t)] = \underbrace{D_x f_0(t, x(t))}_{\text{Jacobian}} h(t)$

$\to$ F_3 similar

$\cdot)$ Invertibility of $d_x F(x_0(t), 0) \in \mathcal{L}(\mathcal{C}^1_T, \mathcal{C}^0_T)$

$$d_x F(x_0(t), \varepsilon)[h(t)] = \frac{d}{dt} h(t) - D_x f_0(t, x_0(t))[h(t)] - \varepsilon D_x f_1(t, x_0(t))[h(t)]$$

$\leadsto$ $d_x F(x_0(t), 0)[h(t)] = \frac{d}{dt} h(t) - D_x f_0(t, x_0(t))[h(t)]$

Invertibility: $\forall g \in \mathcal{C}^0_T$, find $1!$ $h \in \mathcal{C}^1_T$ sol of

$$d_x F(x_0(t), 0)[h(t)] = g(t) \iff \frac{d}{dt} h(t) = D_x f_0(t, x_0(t)) . h(t) + g(t)$$

Put $B(t) := D_x f_0(t, x_0(t))$ linear op

T - periodic

$$\Rightarrow \quad \frac{d}{dt} h(t) = B(t)\, h(t) + g(t)$$

Denote by $M(t) \in \mathrm{Mat}(\mathbb{R}^n \times \mathbb{R}^n)$ the resolvent matrix of $\frac{d}{dt} h(t) = B(t)\, h(t)$, i.e. (homogeneous problem)

$$\begin{cases} \frac{d}{dt} M(t) = B(t)\, M(t) \\ M(0) = \mathbb{I} \end{cases}$$

Then $h(t) = M(t)\, z$ solves $\begin{cases} \frac{d}{dt} h(t) = B(t)\, h(t) \\ h(0) = z \end{cases}$

Now consider inhomogeneous problem:

$$\begin{cases} \frac{d}{dt} h(t) = B(t)\, h(t) + g(t) \\ h(0) = z \end{cases}$$

By variations of constants the solution is

$$h(t) = M(t)\, z + M(t) \int_0^t M^{-1}(\tau)\, g(\tau)\, d\tau$$

Need $h(t)$ to be T-periodic

$$h(T) = M(T)\, z + M(T) \int_0^T M^{-1}(\tau)\, g(\tau)\, d\tau = h(0) = z$$

$$\Rightarrow \quad \underbrace{\left(\mathbb{I} - M(T)\right)}_{\text{need to invert this operator}} z = M(T) \int_0^T M^{-1}(\tau)\, g(\tau)\, d\tau$$

$\mathbb{1} - M(T)$ invertible $\iff$ $1 \notin \sigma(M(T))$

Then $\exists !\ z$ s.t. $\begin{cases} \frac{d}{dt} h = \overbrace{B(t)}^{\in C^0} h + \overset{T}{g}^{\,C^0} \\ h(T) = h(0) \end{cases}$ $\Rightarrow h \in C^1_T$

Thm if $1 \notin \sigma(M(T))$, then $\forall \varepsilon$ small enough, $\exists !$ T periodic sol $x_\varepsilon(t)$ of perturbed eq close to $x_0(t)$

Autonomous ODEs

$\dot{x} = f_0(x) + \varepsilon f_1(x)$, $f_0, f_1 \in C^1(\mathbb{R}^n, \mathbb{R}^n)$

again assume $\dot{x} = f_0(x)$ has T - periodic sol $x_0(t)$

First attempt: apply previous result: check if $1 \notin \sigma(M(T))$

where $M(T)$ is the fundamental matrix $\begin{cases} \frac{d}{dt} M(t) = B(t) M(t) \\ M(0) = \mathbb{1} \end{cases}$

and $B(t) = D_x f(x_0(t))$

Now system is time invariant: $x_\theta(t) := x_0(t+\theta)$ is again a solution $\forall \theta$ (before this was false since system not autonomous)

$\leadsto \frac{d}{dt} x_\theta(t) = f_0(x_\theta(t)) \quad \forall t$

$\leadsto$ take $\partial_\theta \big|_{\theta=0}$: $\frac{d}{dt} \underbrace{\partial_\theta x_\theta(t)\Big|_{\theta=0}}_{\dot{x}_0(t)} = \underbrace{D_x f_0(x_0(t))}_{B(t)} \underbrace{\partial_\theta x_\theta(t)}_{\dot{x}_0(t)}\Big|_{\theta=0}$

So $\dot{x}_0(t)$ solves $\begin{cases} \frac{d}{dt} h = B(t) h \\ \cancel{x} h(0) = \dot{x}_0(0) \end{cases}$

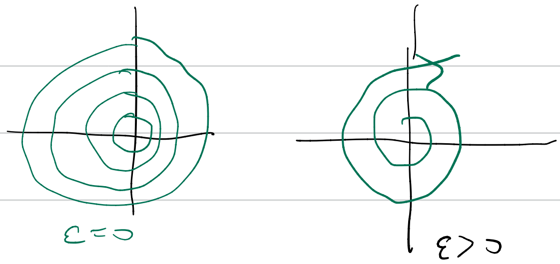
$\rightarrow \quad \dot{x}_0(t) = M(t)\,\dot{x}_0(0) \qquad \forall t$

$$\left.\begin{array}{l} \rightsquigarrow \quad \underbrace{\dot{x}_0(T)}_{\substack{\parallel \ T-\text{periodicity} \\ \dot{x}_0(0)}} = M(T)\,\dot{x}_0(0) \end{array}\right\} \Rightarrow \dot{x}_0(0) \text{ is eigenv. of } M(T) \text{ with eigenvalue } 1$$

$\Rightarrow \quad 1 \in \sigma(M(T))$!

Rem persistence of periodic orbits in autonomous system might fail! There are systems where <u>all</u> periodic orbits disappear

$$\begin{cases} \dot{x} = -y + \varepsilon\, x(x^2+y^2) \\ \dot{y} = x + \varepsilon\, y\,(x^2+y^2) \end{cases}$$



Need more freedom: allow periodic orbits to have a different period T_ε, changing with ε

So we look for solutions which are $T_\varepsilon = \frac{T}{\omega_\varepsilon}$ periodic with $\omega_\varepsilon \to 1$ as $\varepsilon \to 0$:

<u>Problem 1</u> the space $C^1_{T_\varepsilon}$ change with ε! No good for IFT

Time rescaling: look for

$$x_\varepsilon(t) = \tilde{x}_\varepsilon(\omega_\varepsilon t) \qquad \text{with } \tilde{x}_\varepsilon \in C^1_T$$

then x_ε is T_ε-periodic sol

$\Updownarrow$

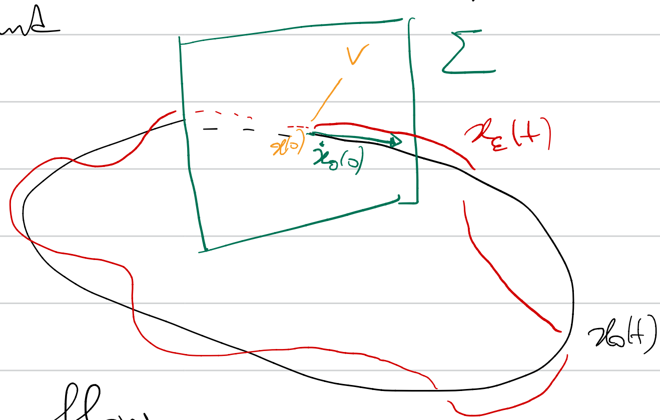
$\tilde{x}_\varepsilon$ is T-periodic sol of

$$\omega_\varepsilon\, \dot{\tilde{x}}_\varepsilon = f_0(\tilde{x}_\varepsilon(t)) + \varepsilon\, f_1(\tilde{x}_\varepsilon(t))$$

Going to look for zeroes of

$$\omega \dot{x} - f_0(x) - \varepsilon f_1(x) = 0$$

Problem 2 Still too much freedom! as in the previous case, we will want to select an initial data of periodic orbit close to $x_0(t)$. But $x_\varepsilon(0)$ and $\dot{x}_\varepsilon(0)$ give rise to same orbit because of translation invariance.



Poincaré section: let

Σ transversal section to the flow

$$\Sigma = \{ v \in \mathbb{R}^n : \langle v - x_0(0), \dot{x}_0(0) \rangle = 0$$

We look for initial datum in Σ: the flow will be transverse to Σ in a small neighbourhood of $x_0(0)$, so in such neighbour any flow line cuts Σ in just 1 point.

Conclusion: look for zeros of

$$F : C^1_T \times \mathbb{R} \times \mathbb{R} \longrightarrow C^0_T \times \mathbb{R}$$

$$(x(t), \omega, \varepsilon) \longrightarrow \begin{pmatrix} \omega \dot{x} - f_0(x(t)) - \varepsilon f_1(x(t)) \\ \langle x(0) - x_0(0), \dot{x}_0(0) \rangle \end{pmatrix}$$

$\circ)$ $F(1, x_0(t), 0) = \begin{pmatrix} \dot{x}_0 - f_0(x_0(t)) \\ \langle x_0(0) - x_0(0), \dot{x}_0(0) \rangle \end{pmatrix} = 0$

$\cdot)$ $F \in C^1$ ✓

$\cdot)$ Invertibility of $\partial_{(x,\omega)} F(1, x_0(t), 0)$

$$(\partial_{(x,\omega)} F)(1, x_0(t), 0)[h, \ell] = \begin{pmatrix} \ell \dot{x}_0(t) + \dot{h} - D_x f_0(x_0(t)) \cdot h(t) \\ \langle h(0), \dot{x}_0(0) \rangle \end{pmatrix}$$

given $\begin{pmatrix} g \\ \alpha \end{pmatrix} \in C^0_T \times \mathbb{R}$, find $(h, \ell) \in C^1_T \times \mathbb{R}$

solving $\begin{cases} \dot{h} = B(t) h - \ell \dot{x}_0(t) + g(t) \\ \langle h(0), \dot{x}_0(0) \rangle = \alpha \end{cases}$

$\underline{1^{st} \text{ eq}}$ by variation of constants: the sol with initial datum $z \in \mathbb{R}^n$ is

$$h(t) = M(t) z + M(t) \int_0^t M^{-1}(\tau) g(\tau) d\tau - \ell M(t) \int_0^t \underbrace{M^{-1}(\tau) \dot{x}_0(\tau)}_{\dot{x}_0(0) \ \forall \tau} d\tau$$

$$\rightsquigarrow h(t) = M(t) z + M(t) \int_0^t M^{-1}(\tau) g(\tau) d\tau - \ell t M(t) \dot{x}_0(0)$$

we want $h(T) = h(0)$

$$\rightarrow z = M(T) z + \underbrace{M(T) \int_0^T M^{-1}(\tau) g(\tau) d\tau}_{b \in \mathbb{R}^n} - \ell T \underbrace{M(T) \dot{x}_0(0)}_{\dot{x}_0(0)}$$

$$\rightsquigarrow (\mathbb{1} - M(T)) z = b - \ell T \dot{x}_0(0) \qquad (*)$$

We know that $1 \in \sigma(M(T))$

ASSUME that 1 is simple eigenvalue,

Then $\mathbb{R}^n = \underbrace{\ker(\mathbb{1} - M(T))}_{\text{span}\langle \dot{x}_0(0)\rangle} \oplus \operatorname{Im}(\mathbb{1} - M(T))$ (exercise!)

$\hookrightarrow \ker(\mathbb{1} - M(T)) \cap \operatorname{Im}(\mathbb{1} - M(T)) = \{0\}$

write $\quad z = a_0 \dot{x}_0(0) + \hat{z}, \quad \hat{z} \in \operatorname{Im}(\mathbb{1} - M(T))$

$\quad b = b_0 \dot{x}_0(0) + \hat{b} \quad \hat{b} \in \operatorname{Im}(\mathbb{1} - M(T))$

Then $(*)$ and $\langle h(0), \dot{x}_0(0)\rangle = \alpha$ become

$$\begin{cases} (\mathbb{1} - M(T))\hat{z} = (b_0 - \Omega T)\dot{x}_0(0) + \hat{b} \\ a_0 \|\dot{x}_0(0)\|^2 + \langle \hat{z}, \dot{x}_0(0)\rangle = \alpha \end{cases}$$

unknown: $\hat{z}, a_0, \Omega$. To solve 1st eq need r.h.s in $\operatorname{Im}(\mathbb{1} - M(T))$

$$\rightsquigarrow \begin{cases} b_0 = \Omega T & \leftarrow \text{select } \Omega \\ (\mathbb{1} - M(T))\hat{z} = \hat{b} & \leftarrow \exists!\ \text{sol } \hat{z} \in \operatorname{Im}(\mathbb{1} - M(T)) \\ a_0 = (\alpha - \langle \hat{z}, \dot{x}_0(0)\rangle)/\|\dot{x}_0(0)\|^2 & \leftarrow \text{select } a_0 \end{cases}$$

($\hat{z}_1, \hat{z}_2$ solve 2nd eq $\Rightarrow$ $\hat{z}_1 - \hat{z}_2 \in \ker(\mathbb{1} - M(T)) \cap \operatorname{Im}(\mathbb{1} - M(T)) = \{0\}$)

Thm (Poincaré continuation theorem) Assume that 1 is a simple eigenv. of $M(T)$. Then $\forall \varepsilon$ small enough, $\exists\, T_\varepsilon$ - periodic sol $x_\varepsilon(t)$ of $\dot{x} = f_0(x) + \varepsilon f_1(x)$ with $x_\varepsilon \to x_0$ in C^1 and $T_\varepsilon \to T$ as $\varepsilon \to 0$